Elemento isoparamétrico de 4 nodos

Curso de Geomecánica Computacional
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Funciones de interpolación

Las funciones de interpolación del elemento isoparamétrico de cuatro nodos son

In[127]:=
```
Clear [ h , ξ ]
h1 = ( 1 - ξ1 ) * ( 1 - ξ2 ) / 4 ;
h2 = ( 1 + ξ1 ) * ( 1 - ξ2 ) / 4 ;
h3 = ( 1 + ξ1 ) * ( 1 + ξ2 ) / 4 ;
h4 = ( 1 - ξ1 ) * ( 1 + ξ2 ) / 4 ;
```

La matriz de interpolación es

In[132]:=
```
Clear[H]
H = (h1  0  h2  0  h3  0  h4  0 ) ;
    (0   h1  0  h2  0  h3  0  h4)
Print [ MatrixForm [ H ] ]
```

$$\begin{pmatrix} \frac{1}{4}(1-\xi_1)(1-\xi_2) & 0 & \frac{1}{4}(1+\xi_1)(1-\xi_2) & 0 & \frac{1}{4}(1+\xi_1)(1+\xi_2) & \cdots \\ 0 & \frac{1}{4}(1-\xi_1)(1-\xi_2) & 0 & \frac{1}{4}(1+\xi_1)(1-\xi_2) & 0 & \frac{1}{4}\cdots \end{pmatrix}$$

Vectores nodales

Los vectores de coordenadas y de desplazamientos nodales son

In[135]:=
```
Clear [ Xn , Un , X , U ]
Xn = Flatten [ Table [ X_{i,j} , { i , 1 , 4 } , { j , 1 , 2 } ] ] ;
Un = Flatten [ Table [ U_{i,j} , { i , 1 , 4 } , { j , 1 , 2 } ] ] ;
Print [ Xn ]
Print [ Un ]
```

$\{X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{4,1}, X_{4,2}\}$

$\{U_{1,1}, U_{1,2}, U_{2,1}, U_{2,2}, U_{3,1}, U_{3,2}, U_{4,1}, U_{4,2}\}$

Coordenadas y desplazamientos interpolados

Las coordenadas y los desplazamientos en cualquier punto del elemento se calculan con las interpolaciones

In[140]:=
```
Clear [ xe , ue ]
xe = H . Xn ;
ue = H . Un ;
Print [ MatrixForm [ xe ] ]
Print [ MatrixForm [ ue ] ]
```

$$\begin{pmatrix} \frac{1}{4}(1-\zeta_1)(1-\zeta_2)X_{1,1}+\frac{1}{4}(1+\zeta_1)(1-\zeta_2)X_{2,1}+\frac{1}{4}(1+\zeta_1)(1+\zeta_2)X_{3,1}+\frac{1}{4}(1-\zeta_1)(1+\zeta_2)X_{4,1} \\ \frac{1}{4}(1-\zeta_1)(1-\zeta_2)X_{1,2}+\frac{1}{4}(1+\zeta_1)(1-\zeta_2)X_{2,2}+\frac{1}{4}(1+\zeta_1)(1+\zeta_2)X_{3,2}+\frac{1}{4}(1-\zeta_1)(1+\zeta_2)X_{4,2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4}(1-\zeta_1)(1-\zeta_2)U_{1,1}+\frac{1}{4}(1+\zeta_1)(1-\zeta_2)U_{2,1}+\frac{1}{4}(1+\zeta_1)(1+\zeta_2)U_{3,1}+\frac{1}{4}(1-\zeta_1)(1+\zeta_2)U_{4,1} \\ \frac{1}{4}(1-\zeta_1)(1-\zeta_2)U_{1,2}+\frac{1}{4}(1+\zeta_1)(1-\zeta_2)U_{2,2}+\frac{1}{4}(1+\zeta_1)(1+\zeta_2)U_{3,2}+\frac{1}{4}(1-\zeta_1)(1+\zeta_2)U_{4,2} \end{pmatrix}$$

Jacobiano

El Jacobiano de la transformación es

In[145]:=
```
Clear [ J ]
J = Table [ ∂ξi xe [ [ j ] ] , { i , 1 , 2 } , { j , 1 , 2 } ] // FullSimplify ;
Print [ MatrixForm [ J ] ]
```

$$\begin{matrix} \frac{1}{4}\big((-1+\zeta_2)X_{1,1}-(-1+\zeta_2)X_{2,1}+(1+\zeta_2)(X_{3,1}-X_{4,1})\big) & \frac{1}{4}\big((-1+\zeta_2)X_{1,2}-(-1+\zeta_2)X_{2,2}+(1+\zeta_2)\,(\\ \frac{1}{4}\big((-1+\zeta_1)X_{1,1}-(1+\zeta_1)(X_{2,1}-X_{3,1})-(-1+\zeta_1)X_{4,1}\big) & \frac{1}{4}\big((-1+\zeta_1)X_{1,2}-(1+\zeta_1)(X_{2,2}-X_{3,2})-(- \end{matrix}$$

Matriz de interpolación de deformaciones

La matriz de interpolación de deformaciones contiene las derivadas de H ordenadas de manera que puedan multiplicarse por el vector de desplazamientos nodales. La secuencia de construcción puede ser:

1- Se genera una tabla con todas las derivadas parciales

In[148]:=
```
Clear [ B1 ]
B1 = Table [ ∂ξj hi , { j , 1 , 2 } , { i , 1 , 4 } ] ;
Print [ MatrixForm [ B1 ] ]
```

$$\begin{pmatrix} \frac{1}{4}(-1+\zeta_2) & \frac{1}{4}(1-\zeta_2) & \frac{1}{4}(1+\zeta_2) & \frac{1}{4}(-1-\zeta_2) \\ \frac{1}{4}(-1+\zeta_1) & \frac{1}{4}(-1-\zeta_1) & \frac{1}{4}(1+\zeta_1) & \frac{1}{4}(1-\zeta_1) \end{pmatrix}$$

2- Se la premultiplica por la inversa del Jacobiano para hacer el cambio de coordenadas

In[151]:=
```
Clear [ B2 ]
B2 = Inverse [ J ] . B1 ;
Print [ MatrixForm [  B2 ] ]
```

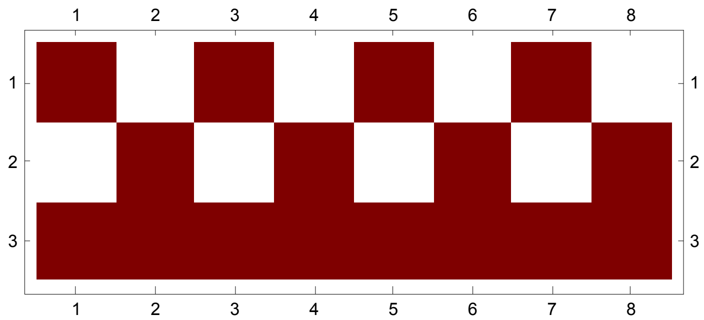
$$\begin{matrix} \frac{(-1+\zeta_1)\,(-(-1+\zeta_2)X_{1,2}+(-1+\zeta_2)X_{2,2}-(1+\zeta_2)(X_{3,2}-X_{4,2}))}{16\,(-\frac{1}{16}((-1+\zeta_1)X_{1,1}-(1+\zeta_1)(X_{2,1}-X_{3,1})-(-1+\zeta_1)X_{4,1})((-1+\zeta_2)X_{1,2}-(-1+\zeta_2)X_{2,2}+(1+\zeta_2)(X_{3,2}-X_{4,2}))+\frac{1}{16}((-1+\zeta_2)X_{1,1}-(-1+\zeta_2)X_{2,1}+(1+\zeta} \\ \frac{(-1+\zeta_1)\,((-1+\zeta_2)X_{1,1}-(-1+\zeta_2)X_{2,1}+(1+\zeta_2)(X_{3,1}-X_{4,1}))}{16\,(-\frac{1}{16}((-1+\zeta_1)X_{1,1}-(1+\zeta_1)(X_{2,1}-X_{3,1})-(-1+\zeta_1)X_{4,1})((-1+\zeta_2)X_{1,2}-(-1+\zeta_2)X_{2,2}+(1+\zeta_2)(X_{3,2}-X_{4,2}))+\frac{1}{16}((-1+\zeta_2)X_{1,1}-(-1+\zeta_2)X_{2,1}+(1+\zeta} \end{matrix}$$

3- Se construye la matriz B de interpolación de deformaciones, cuya forma es

$$B = \begin{pmatrix} B2_{11} & 0 & B2_{12} & 0 & B2_{13} & 0 & B2_{14} & 0 \\ 0 & B2_{21} & 0 & B2_{22} & 0 & B2_{23} & 0 & B2_{24} \\ B2_{21} & B2_{11} & B2_{22} & B2_{12} & B2_{23} & B2_{13} & B2_{24} & B2_{14} \end{pmatrix};$$

In[154]:=
```
Clear [ B ]
B = {
 Riffle [ B2 [ [ 1 ] ] , { 0 , 0 , 0 , 0 } ] ,
 Riffle [ { 0 , 0 , 0 , 0 } , B2 [ [ 2 ] ] ] ,
 Riffle [ B2 [ [ 2 ] ] , B2 [ [ 1 ] ] ] } ;
Print [ MatrixForm [ B ] ]
Print [ MatrixPlot [ B ] ]
```

$$\left(\begin{array}{l} \frac{(-1+\zeta_1)\ (-(-1+\zeta_2)\ X_{1,2}+(-1+\zeta_2)\ X_{2,2}-(1+\zeta_2)\ (X_{3,2}-X_{4,2}))}{16\ \left(-\frac{1}{16}\ ((-1+\zeta_1)\ X_{1,1}-(1+\zeta_1)\ (X_{2,1}-X_{3,1})-(-1+\zeta_1)\ X_{4,1})\ ((-1+\zeta_2)\ X_{1,2}-(-1+\zeta_2)\ X_{2,2}+(1+\zeta_2)\ (X_{3,2}-X_{4,2}))+\frac{1}{16}\ ((-1+\zeta_2)\ X_{1,1}-(-1+\zeta_2)\ X_{2,1}+(1+\zeta \right.} \\ \\ \frac{(-1+\zeta_1)\ ((-1+\zeta_2)\ X_{1,1}-(-1+\zeta_2)\ X_{2,1}+(1+\zeta_2)\ (X_{3,1}-X_{4,1}))}{16\ \left(-\frac{1}{16}\ ((-1+\zeta_1)\ X_{1,1}-(1+\zeta_1)\ (X_{2,1}-X_{3,1})-(-1+\zeta_1)\ X_{4,1})\ ((-1+\zeta_2)\ X_{1,2}-(-1+\zeta_2)\ X_{2,2}+(1+\zeta_2)\ (X_{3,2}-X_{4,2}))+\frac{1}{16}\ ((-1+\zeta_2)\ X_{1,1}-(-1+\zeta_2)\ X_{2,1}+(1+\zeta \right.} \end{array}\right.$$



Ejemplo I

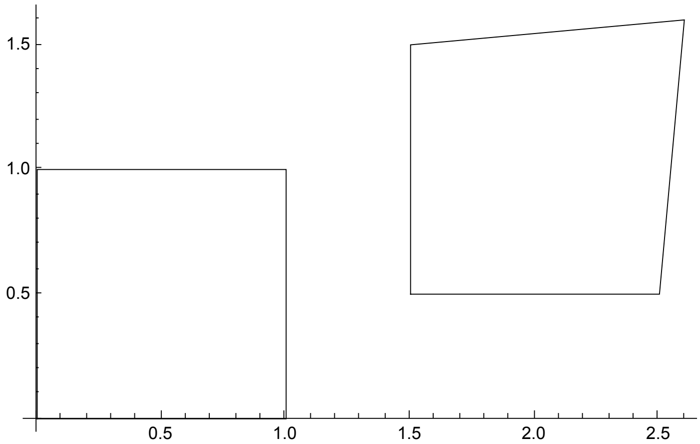
La formulación se aplica a un elemento de 1x1

In[158]:=
```
Clear [ Xnodos , coords , conec , pos1 ]
Xnodos = { 0 , 0 , 1 , 0 , 1 , 1 , 0 , 1 } ;
coords = Partition [ Xnodos , 2 ] ;
conec = { { 1 , 2 , 3 , 4 , 1 } } ;
pos1 = Graphics [ GraphicsComplex [ coords , Line [ conec ] ] , Axes → True ] ;
```

Se impone un desplazamiento {1,0.5} y un desplazamiento adicional {0.1,0.1} en su nodo 3

In[163]:=
```
Clear [  Ux , Uy , ΔU3 , Unodos , coords2 , pos2 ]
Ux = 1.5 ; Uy = 0.5 ; ΔU3 = 0.1 ;
Unodos = Ux * { 1 , 0 , 1 , 0 , 1 , 0 , 1 , 0 } + Uy * { 0 , 1 , 0 , 1 , 0 , 1 , 0 , 1
coords2 = Partition [ Xnodos + Unodos , 2 ] ;
pos2 = Graphics [ GraphicsComplex [ coords2 , Line [ conec ] ] , Axes → True ] ;
Show [ pos1 , pos2 ]
```

Out[168]=



La regla permite reemplazar las variables por sus valores sin perder la definición original

In[169]:=
```
rulecoord = Table [ Xn [ [ i ] ] → Xnodos [ [ i ] ] , { i , 1 , 8 } ] ;
ruledespl = Table [ Un [ [ i ] ] → Unodos [ [ i ] ] , { i , 1 , 8 } ] ;
```

Se calculan los desplazamientos y deformaciones

In[171]:=
```
u = ue /. ruledespl
ϵ = B . Unodos /. ruledespl /. rulecoord
```

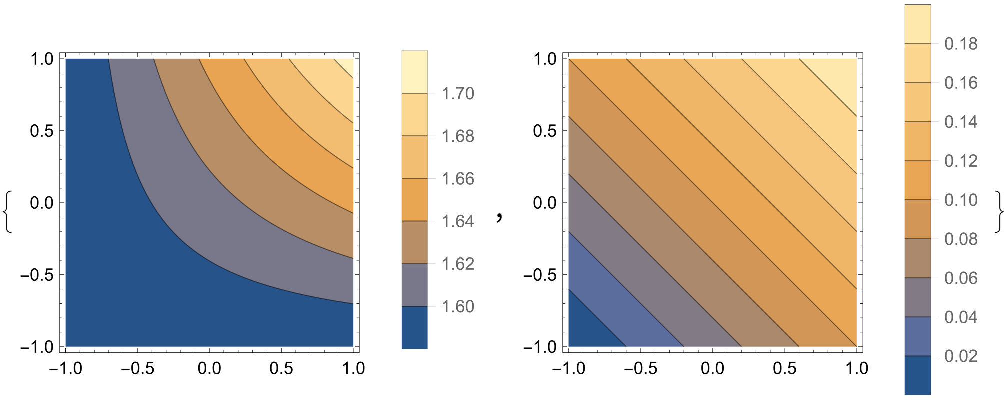
Out[171]= $\{0.375\,(1-\zeta_1)\,(1-\zeta_2)+0.375\,(1+\zeta_1)\,(1-\zeta_2)+0.375\,(1-\zeta_1)\,(1+\zeta_2)+0.4\,(1+\zeta_1)\,(1+\zeta_2),\ 0.125\,(1-\zeta_1)\,(1-\zeta_2)+0.125\,(1+\zeta_1)\,(1-\zeta_2)+0.125\,(1-\zeta_1)\,(1+\zeta_2)+0.15\,(1+\zeta_1)\,(1+\zeta_2)\}$

Out[172]= $\{0.+0.75\,(-1-\zeta_2)+0.75\,(1-\zeta_2)+0.75\,(-1+\zeta_2)+0.8\,(1+\zeta_2),\ 0.+0.25\,(-1-\zeta_1)+0.25\,(1-\zeta_1)+0.25\,(-1+\zeta_1)+0.3\,(1+\zeta_1),\ 0.75\,(-1-\zeta_1)+0.75\,(1-\zeta_1)+0.75\,(-1+\zeta_1)+0.8\,(1+\zeta_1)+0.25\,(-1-\zeta_2)+0.25\,(1-\zeta_2)+0.25\,(-1+\zeta_2)+0.3\,(1+\zeta_2)\}$

Se dibuja la norma del desplazamiento, la deformación volumétrica

In[173]:=
```
{ContourPlot[Norm[ue /. ruledespl], {ξ1, -1, 1}, {ξ2, -1, 1}, PlotLege
ContourPlot[ϵ[[1]] + ϵ[[2]], {ξ1, -1, 1}, {ξ2, -1, 1}, PlotLeger
```

Out[173]=



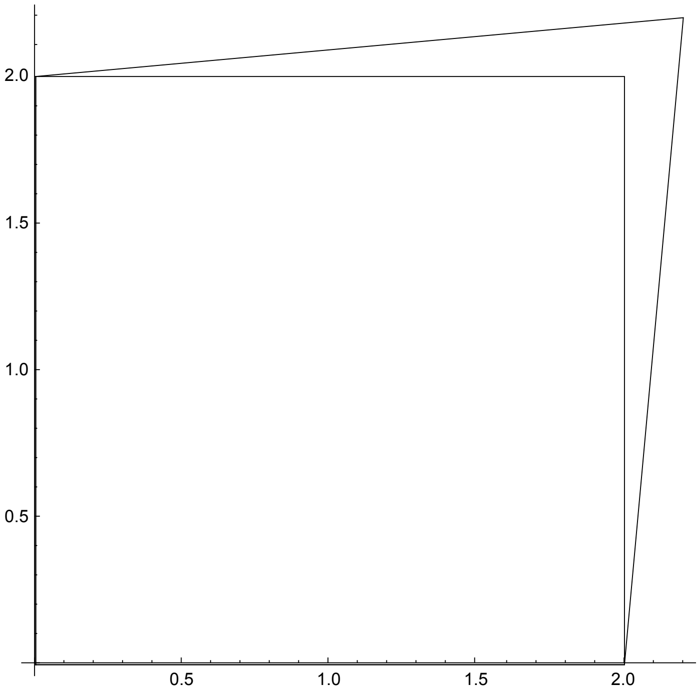
Ejemplo 2

Desplazamiento {0.2,0.2} en el nodo 3 de un elemento de 2 x 2. Todo lo demás igual. El desplazamiento es distinto, pero la deformación es la misma que en el caso anterior. Este ejercicio demuestra que la formulación isoparamétrica [0,1] transformada por el Jacobiano es equivalente a una interpolación directa en [x,y]

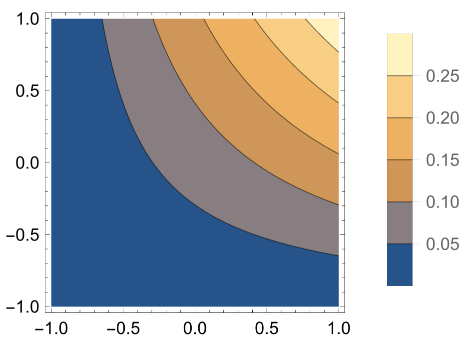
In[174]:=

```
Xnodos = {0, 0, 2, 0, 2, 2, 0, 2};
Unodos = {0, 0, 0, 0, 0.2, 0.2, 0, 0};
rulecoord = Table[Xn[[i]] → Xnodos[[i]], {i, 1, 8}];
ruledespl = Table[Un[[i]] → Unodos[[i]], {i, 1, 8}];
coords = Partition[Xnodos, 2];
conec = {{1, 2, 3, 4, 1}};
pos1 = Graphics[GraphicsComplex[coords, Line[conec]], Axes → True];
coords2 = Partition[Xnodos + Unodos, 2];
pos2 = Graphics[GraphicsComplex[coords2, Line[conec]], Axes → True];
Show[pos1, pos2]
u = ue /. ruledespl;
ϵ = B . Unodos /. ruledespl /. rulecoord;
{ContourPlot[Norm[ue /. ruledespl], {ξ1, -1, 1}, {ξ2, -1, 1}, PlotLege
ContourPlot[ϵ[[1]] + ϵ[[2]], {ξ1, -1, 1}, {ξ2, -1, 1}, PlotLeger
```

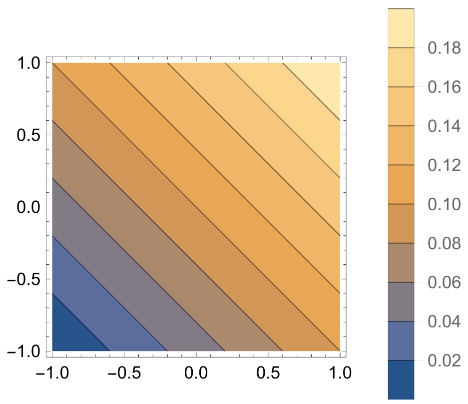
Out[183]=



Out[186]= {



,



}